Lezione 3

Assistente

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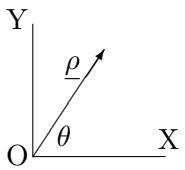
1 Ripasso



$x = \rho \cos\theta$
$y = \rho \sin\theta$
$\rho = \sqrt{x^2 + y^2}$
$\theta = \arctan\left(\frac{y}{x}\right)$

2 Conversione di coordinate

$P(x, y, z) \rightarrow P(\rho, \phi, \theta)$

ϕ (Phi) = angolo zenitale = latitudine
θ (Theta) = angolo azimutale = longitudine

2.1 Da polari a cartesiane

$z = \rho \cos\phi$

$\frac{z}{\rho} = \cos\phi \qquad \frac{\rho_{x,y}}{\rho} = \sin\phi \qquad \rho_{x,y} = \rho \sin\phi$

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$

2.2 Da cartesiane a polari

$\theta = \arctan(\frac{y}{x})$
$\rho = \sqrt{x^2 + y^2 + z^2}$
$\phi = \arccos\left(\frac{z}{\rho}\right)$

3 Esercizi

3.1 Da cartesiane a polari

$P = \left(\frac{\sqrt{3}}{3}; \frac{\sqrt{3}}{3}; \frac{\sqrt{3}}{3}\right)$

$\rho = \sqrt{\frac{3}{9} + \frac{3}{9} + \frac{3}{9}} = \sqrt{\frac{9}{9}} = 1$

$\phi = \arccos(\frac{\sqrt{3}}{3})$

$\theta = \arctan(1) = \frac{\pi}{4}$

3.2

$P = (0, 0, 3)$

$\rho = \sqrt{9} = 3$

$\phi = \arccos\left(\frac{3}{3}\right) = \arccos(1) = 0$

$\theta =$ non esiste (il punto si trova sull'asse Z)

3.3

$P = (0, 1, 0)$

$\rho = 1$

$\phi = \arccos\left(\frac{0}{1}\right) = \frac{\pi}{2}$

$\theta = \arctan\left(\frac{1}{0}\right) = \frac{\pi}{2}$

3.4 Da polari a cartesiane

$P = \left(1, \frac{\pi}{2}, \frac{3}{2}\pi\right)$ $\rightarrow$ ricordarsi che sono espresse nell'ordine (ρ, θ, ϕ)

$z = 1 \cdot \cos\left(\frac{3}{2}\pi\right) = 1 \cdot 0 = 0 \qquad \Rightarrow \qquad \sin\left(\frac{3}{2}\pi\right) = -1$

$x = 1 \cdot \sin\left(\frac{3}{2}\pi\right) \cos(\frac{\pi}{2}) = 0$

$y = 1 \cdot \sin\left(\frac{3}{2}\pi\right) \sin(\frac{\pi}{2}) = -1$

3.5

$P = \left(\frac{1}{2}, \frac{\pi}{3}, \pi\right)$

$z = \frac{1}{2} \cos \pi = -\frac{1}{2}$

$x = \frac{1}{2} \sin(\pi) \cos\left(\frac{\pi}{3}\right) = 0$

$y = \frac{1}{2} \sin(\pi) \sin\left(\frac{\pi}{3}\right) = 0$

4 Somma di vettori

4.1

$$\begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 2 \end{bmatrix}$$

4.2

$$\begin{bmatrix} -1 \\ 4 \\ 9 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 13 \end{bmatrix}$$

4.3

$$3 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + (-5) \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix} + \begin{bmatrix} 10 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ -2 \\ 14 \end{bmatrix}$$

5 Prodotto scalare di matrici

Il prodotto scalare è la somma dei prodotti delle componenti omologhe.

5.1

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 3 + 2 - 2 = 3$$

5.2

$$\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ -2 \\ 4 \end{bmatrix} = 6 - 10 + 4 = 0$$

5.3

$$2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot 3 \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix} = -12 + 0 + 18 = 6$$

5.4

$(a\underline{\mathbf{v}}) \cdot (b\underline{\mathbf{u}}) = ab(\underline{\mathbf{v}} \cdot \underline{\mathbf{u}})$

$\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} = |\underline{\mathbf{u}}|\,|\underline{\mathbf{v}}| \cos\theta$

$\frac{\underline{\mathbf{u}} \cdot \underline{\mathbf{v}}}{|\underline{\mathbf{w}}||\underline{\mathbf{v}}|} = \cos\theta \qquad \Rightarrow \qquad \arccos\left(\frac{\underline{\mathbf{u}} \cdot \underline{\mathbf{v}}}{|\underline{\mathbf{u}}||\underline{\mathbf{v}}|}\right) = \theta$

5.5

$\underline{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \underline{\mathbf{v}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} = 0$
$|\underline{\mathbf{u}}| = \sqrt{1^2 + 1^2} = \sqrt{2}$
$|\underline{\mathbf{v}}| = 1$

$\arccos(0) = \frac{\pi}{2} \qquad \Rightarrow \qquad$ Due vettori con prodotto scalare nullo sono $\perp$

5.6 Esercizi

$\underline{\mathbf{u}} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} \qquad \underline{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} = 2 + 8 = 10$
$|\underline{\mathbf{u}}| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$
$|\underline{\mathbf{v}}| = \sqrt{1+4} = \sqrt{5}$

$\arccos \frac{\underline{\mathbf{u}} \cdot \underline{\mathbf{v}}}{|\underline{\mathbf{u}}||\underline{\mathbf{v}}|} = \arccos\left(\frac{10}{2\sqrt{5}\sqrt{5}}\right) = \arccos\left(\frac{10}{10}\right) = \arccos(1) = 0$

5.7

$\underline{\mathbf{u}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{\mathbf{v}} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}$

$\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} = \frac{\sqrt{3}}{2}$
$|\underline{\mathbf{u}}| = 1$
$|\underline{\mathbf{v}}| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\arccos \frac{\underline{\mathbf{u}} \cdot \underline{\mathbf{v}}}{|\underline{\mathbf{u}}| \cdot |\underline{\mathbf{v}}|} = \frac{\sqrt{3}/2}{1 * 1} = \frac{\sqrt{3}}{2}$

$\arccos \frac{\sqrt{3}}{2} = \frac{\pi}{6} \qquad$ quale angolo ha per $\cos \frac{\sqrt{3}}{2}$? $\rightarrow \frac{\pi}{6}$, cioé $30°$.

5.8

$\underline{\mathbf{u}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \qquad \underline{\mathbf{v}} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$

$\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} = 6$
$|\underline{\mathbf{u}}| = 3$
$|\underline{\mathbf{v}}| = \sqrt{4+4+0} = 2\sqrt{2}$
$\theta = \arccos\left(\frac{6}{6\sqrt{2}}\right) = \arccos \frac{1}{\sqrt{2}} = \arccos \frac{\sqrt{2}}{2}$

5.9

$$\underline{\mathbf{u}} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \qquad \underline{\mathbf{v}} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$\underline{\mathbf{u}} \cdot \underline{\mathbf{v}} = 6$
$|\underline{\mathbf{u}}| = 3$
$|\underline{\mathbf{v}}| = \sqrt{4+4+0} = 2\sqrt{2}$
... (come sopra)

5.10

$$\underline{\mathbf{v_1}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{\mathbf{v_2}} = \begin{bmatrix} \frac{3}{2} \\ \frac{3\sqrt{3}}{2} \\ 0 \end{bmatrix}$$

$\left|\underline{\mathbf{v_1}}\right| = 1$
$\left|\underline{\mathbf{v_2}}\right| = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$

$\underline{\mathbf{v_1}} \cdot \underline{\mathbf{v_2}} = \frac{3}{2}$

$\theta = \arccos \frac{\frac{3}{2}}{3} = \arccos\left(\frac{3}{2} \cdot \frac{1}{3}\right) = \arccos\left(\frac{1}{2}\right)$

$\theta = \pm\frac{\pi}{3}$ tengo solo il "+" per la natura dei vettori dati $\quad \Rightarrow \quad \theta = \frac{\pi}{3}$

$l = \left|\underline{\mathbf{v_2}}\right| \cos\theta = 3 \cdot \frac{1}{2} = \frac{3}{2} \qquad$ (l = proiezione di $\underline{\mathbf{v_2}}$ su $\underline{\mathbf{v_1}}$)